(Chapter 8)

100 POINTS

NAME: ____________________

** NO NOTES - NO CALCULATORS**

Find the value of the following improper integrals. Be sure to use all appropriate notation.

(1) $\int_0^4 \frac{1}{(x-2)^3}\,dx$

unbounded at $x=2$

$\int_0^4 \frac{1}{(x-2)^3}dx = \int_0^2 \frac{1}{(x-2)^3}dx + \int_2^4 \frac{1}{(x-2)^3}dx$

$\int_0^2 \frac{1}{(x-2)^3}dx = \lim_{b\to 2^-} \int_0^b \frac{1}{(x-2)^3}dx$

$= \lim_{b\to 2^-} -\frac{1}{(x-2)^2}\Big]_0^b$

$\lim_{b\to 2^-} -\frac{1}{(b-2)^2} + \frac{1}{4}$

$= -\infty$

diverges

(2) $\int_1^\infty \frac{1}{\sqrt{x}\,(x+4)}\,dx$

$\int \frac{1}{\sqrt{x}(x+4)}dx$

$u = \sqrt{x}$
$u^2 = x$
$2u\,du = dx$

$= \int \frac{2u}{u(u^2+4)}du$

$= \int \frac{2}{u^2+4}du$

$= 2 \quad \frac{1}{2}\tan^{-1}\frac{u}{2} + c$

$= \tan^{-1}\frac{\sqrt{x}}{2} + c$

So

$\int_1^\infty \frac{1}{\sqrt{x}(x+4)}dx = \lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}(x+4)}dx$

$= \lim_{b\to\infty} \tan^{-1}\frac{\sqrt{x}}{2}\Big]_1^b$

$= \lim_{b\to\infty}\left[\tan^{-1}\left(\frac{\sqrt{b}}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right]$

$\frac{\pi}{2} - \tan^{-1}\frac{1}{2}$

(3) (a) Use Simpson's Rule with n=10 to approximate the area under the curve $y = e^{-x^2}$, $0 \le x \le 1$.
* Use your calculator efficiently to prevent round-off error.
(b) Estimate the error involved in the above approximation.
(c) If you want to guarantee that the Simpson's Rule approximation is accurate to within 0.00001, how large must n be?.

a) $A = \int_0^1 e^{-x^2}\,dx \qquad \Delta x = \frac{b-a}{n} = \frac{1}{10} \qquad f(x) = e^{-x^2}$

$\approx \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + 2y_8 + 4y_9 + y_{10}\right]$

$= \frac{1}{30}\Big[f(0) + 4f(.1) + 2f(.2) + 4f(.3) + 2f(.4) + 4f(.5) + 2f(.6) + 4f(.7) + 2f(.8) + 4f(.9) + f(1)\Big]$

$= \frac{1}{30}\left[1 + 4e^{-.01} + 2e^{-.04} + 4e^{-.09} + 2e^{-.16} + 4e^{-.25} + 2e^{-.36} + 4e^{-.49} + 2e^{-.64} + 4e^{-.81} + e^{-1}\right]$

≈ 0.746824948

b) Error Bound for Simpson's Rule

$|E_S| \le \frac{J(b-a)^5}{180n^4}$ where $|f^{(4)}(x)| < J$ $\qquad f^{(4)}(x) = \underbrace{4e^{-x^2}(4x^4 - 12x^2 + 3)}_{\text{decreasing on } [0,1]} \le 12$

So max at $x = 0$

$\le \frac{12(1)^5}{180n^4}$

$= \frac{2}{30n^4}$

When $n = 10$, $|E_S| \le \frac{2}{30(10^4)} \approx .00000667$

c) Want $|E_S| \le .00001$

$\frac{2}{30n^4} < .00001$

$\frac{30n^4}{2} > 100{,}000$

$n^4 > 6666.667$

$n > 9.03$

$\boxed{n = 10}$

FOR PROBLEMS 5 - 13, INTEGRATE AND SIMPLIFY

(5) $\int \sin^{3/2} x \cos^3 x\, dx$

$\int \sin^{3/2}x\,(\cos^2 x)\cos x\,dx$

$\int \sin^{3/2}x\,(1 - \sin^2 x)\cos x\,dx \qquad u = \sin x,\ du = \cos x\,dx$

$\int u^{3/2}(1-u^2)\,du = \int \left(u^{3/2} - u^{7/2}\right)du = \frac{2}{5}u^{5/2} - \frac{2}{9}u^{9/2} + C$

$= \frac{2}{5}\sin^{5/2}x - \frac{2}{9}\sin^{9/2}x + C$

(6) $\int \frac{4x+1}{2x^2+x-10}dx = \int \frac{4x+1}{(2x+5)(x-2)} = \int \left(\frac{2}{2x+5} + \frac{1}{x-2}\right)dx$

$u=2x+5$, $du=2dx$ $\qquad$ $u=x-2$, $du=dx$

$= \ln|2x+5| + \ln|x-2| + C$

OR more easily...

$u = 2x^2+x-10$

$du = 4x+1$

$\int \frac{1}{u}du = \ln|2x^2+x-10| + C$

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(7) $\int \sqrt{x}\ln x\,dx$ $\qquad$ $u = \ln x$ $\quad dv = \sqrt{x}\,dx$ $\quad$ by parts

$du = \frac{1}{x}dx$ $\quad V = \frac{2}{3}x^{3/2}$

$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{1/2}dx$

$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$

(8) $\int \frac{1}{1+\sqrt[3]{x}}\,dx$

$u = \sqrt[3]{x}$

$u^3 = x$

$3u^2\,du = dx$

$\int \frac{3u^2\,du}{1+u}$

divide:
$$\begin{array}{r} 3u-3 \\ u+1 \overline{)\,3u^2} \\ -(3u^2+3u) \\ \hline -3u \\ -(-3u-3) \\ \hline 3 \end{array}$$

$\int \left(3u-3+\frac{3}{u+1}\right)du$

$= \frac{3}{2}u^2 - 3u + 3\ln|u+1| + C$

$= \frac{3}{2}x^{2/3} - 3x^{1/3} + 3\ln|x^{1/3}+1| + C$

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(9) $\int \frac{dx}{x^2\sqrt{x^2-16}}$

$x = 4\sec\theta$

$dx = 4\sec\theta\tan\theta\,d\theta$

$\theta \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}]$, $\tan\theta > 0$

$\int \frac{4\sec\theta\tan\theta\,d\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}}$

$\sqrt{16\sec^2\theta-16} = 4\sqrt{\sec^2\theta-1}$

$= 4\sqrt{\tan^2\theta}$

$= 4|\tan\theta|$

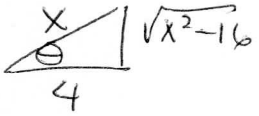
$= 4\tan\theta$

$= \frac{4}{16}\int \frac{\sec\theta\tan\theta}{\sec^2\theta\cdot 4\tan\theta}\,d\theta$

$= \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta$

$= \frac{1}{16}\int \cos\theta\,d\theta$

$\frac{x}{4} = \sec\theta$



$= \frac{1}{16}\sin\theta + C$

$= \frac{1}{16}\frac{\sqrt{x^2-16}}{x} + C$

(10) $\int x^2 \cos(3x)\,dx$

$u = x^2 \qquad dv = \cos 3x\,dx$
$du = 2x\,dx \qquad v = \frac{1}{3}\sin 3x$

$$\int x^2\cos(3x)\,dx = \frac{1}{3}x^2\sin 3x - \frac{2}{3}\underbrace{\int x\sin 3x\,dx}$$

$\bar{u} = x \qquad d\bar{v} = \sin 3x\,dx$
$d\bar{u} = dx \qquad \bar{v} = -\frac{1}{3}\cos 3x$

$$= \frac{1}{3}x^2\sin 3x - \frac{2}{3}\left[-\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx\right]$$

$$= \frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{9}\cdot\frac{1}{3}\sin 3x + C$$

$$= \frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$$

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(11) $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$

$$= \int \frac{x}{\sqrt{4-(x+1)^2}} \qquad u = x+1, \quad du = dx$$

$$= \int \frac{u-1}{\sqrt{4-u^2}}\,du$$

$$= \int \frac{u}{\sqrt{4-u^2}}\,du - \int \frac{1}{\sqrt{4-u^2}}\,du$$

trig. subst. or formula $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$

$\sin^{-1}\frac{u}{2}$

$\bar{u}$-subst

$\bar{u} = 4-u^2$

$d\bar{u} = -2u\,du$

$-\frac{1}{2}\int \bar{u}^{-1/2}\,d\bar{u}$

$-\bar{u}^{1/2}$

$-\sqrt{4-u^2}$

$-\sqrt{3-2x-x^2} \quad - \quad \sin^{-1}\left(\frac{x+1}{2}\right) + C$

(12) $\int \frac{5x^3-3x^2+7x-3}{(x^2+1)^2}dx$

$$\frac{5x^3-3x^2+7x-3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} = \frac{(Ax+B)(x^2+1)+Cx+D}{(x^2+1)^2}$$

$\Rightarrow 5x^3-3x^2+7x-3 = Ax^3+Bx^2+(A+C)x + B+D$

$\Rightarrow A=5 \quad A+C=7 \rightarrow C=2$

$B=-3 \quad B+D=-3 \Rightarrow D=0$

So integral belomes:

$$\int\left(\frac{5x-3}{x^2+1}+\frac{2x}{(x^2+1)^2}\right)dx$$

$$= 5\int\frac{x}{x^2+1}dx - 3\int\frac{1}{x^2+1}dx + \int\frac{2x}{(x^2+1)^2}dx$$

$u=x^2+1 \qquad\qquad u=x^2+1$

$\vdots$

$$= \frac{5}{2}\ln(x^2+1) - 3\tan^{-1}x - \frac{1}{x^2+1} + C$$

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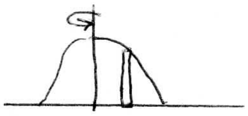
(13) The region under the curve y = $\cos^2 x$, $0 \le x \le \frac{\pi}{2}$, is rotated about the y axis. Find the volume of the resulting solid.



$$V = \int_0^{\pi/2} 2\pi x \cos^2 x \, dx$$

$\int x\cos^2 x\, dx$ by parts $\quad u=x \quad dv=\cos^2 x\, dx$
$\quad du=dx \quad v=\frac{1}{2}x+\frac{1}{4}\sin 2x$

$$= \frac{1}{2}x^2+\frac{1}{4}x\sin 2x - \int\left(\frac{1}{2}x+\frac{1}{4}\sin 2x\right)dx$$

$$= \frac{1}{2}x^2+\frac{1}{4}x\sin 2x - \frac{1}{4}x^2+\frac{1}{8}\cos 2x + c$$

$$= \frac{x^2}{4}+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x + c$$

So $\quad V = 2\pi\left[\frac{x^2}{4}+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x\right]_0^{\pi/2}$

$$V = 2\pi\left[\frac{\pi^2}{16}-\frac{1}{8}-\left(\frac{1}{8}\right)\right] = 2\pi\left[\frac{\pi^2}{16}-\frac{2}{8}\right] = \pi\left[\frac{\pi^2-4}{8}\right]$$