100 POINTS

NAME: ____________

** No credit given unless detailed work is shown. Non-graphing calculators are allowed.**

FILL IN THE BLANK WITH THE MOST APPROPRIATE WORD OR SYMBOL. (2 points each)

(1) The fourth partial sum, S4, of the series $\sum_{n=1}^{\infty} n!$ is $\underline{33}$ $\quad S_4 = 1! + 2! + 3! + 4!$

(2) Find the general term of the sequence $-\frac{3}{4}, \frac{5}{8}, -\frac{7}{16}, \frac{9}{32}, \ldots$ $\underline{\frac{(-1)^n(2n+1)}{2^{n+1}}}$

(3) Determine whether the *sequence* converges or diverges: $\left\{\frac{2n}{5n+1}\right\}$ $\underline{\text{Converges}}$ $\quad \lim_{n\to\infty} \frac{2n}{5n+1} = \frac{2}{5}$

(5) TRUE OR FALSE: If $\sum_{k=1}^{\infty} a_k$ diverges then $\lim_{k\to\infty} a_k \neq 0$. $\underline{\text{false}}$ $\quad$ Consider $\sum \frac{1}{n}$

(5) TRUE OR FALSE: If $0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges $\underline{\text{True}}$

(6) The following series satisfies the conditions of the Alternating Series Test. How many terms would need to be added to approximate the sum of the series with an error less than 10^{-2} in magnitude? (7 points)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{3/2}}$$

need $b_{n+1} < 10^{-2}$

$$\frac{1}{(n+1)^{3/2}} < \frac{1}{100}$$

$$(n+1)^{3/2} > 100$$

$$n > 100^{2/3} - 1 \approx 20.5$$

21 terms

In problems 7 & 8, find the (exact) sum. Show steps in detail. (9 points each)

(7) $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$$S_n = \underbrace{\left(1 - \cancel{\tfrac{1}{3}}\right)}_{n=1} + \underbrace{\left(\tfrac{1}{2} - \cancel{\tfrac{1}{4}}\right)}_{n=2} + \underbrace{\left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right)}_{n=3} + \cdots +$$

$$\cdots + \underbrace{\left(\cancel{\tfrac{1}{n-2}} - \cancel{\tfrac{1}{n}}\right)}_{n-2} + \underbrace{\left(\cancel{\tfrac{1}{n-1}} - \tfrac{1}{n+1}\right)}_{n-1} + \underbrace{\left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+2}\right)}_{n}$$

$$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$S = \lim_{n\to\infty} S_n = \frac{3}{2}$$

(8) $\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{4^n} = (-3) +$

$$\sum_{n=0}^{\infty} \frac{(-3)(-3)^n}{4^n} = -3 \sum_{n=0}^{\infty} \left(\frac{-3}{4}\right)^n$$

$a = 1$

$r = -\frac{3}{4}$

$$S = \frac{a}{1-r} = \frac{1}{1 - -3/4} = \frac{4}{7}$$

so

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}}{4^n} = -3\left(\frac{4}{7}\right) = \frac{-12}{7}$$

(9) For each of the following series, determine the convergence. (Classify as absolute or conditional if applicable) If convergence is conditional, show how absolute convergence was ruled out. Name any test(s) used and verify test applies to given series.

(9 points each)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^3+5n^2}{n^4-3n^3+2}$

Limit comparison test, compare to $\sum \frac{1}{n}$ which diverges (harmonic)

$$C = \lim_{n\to\infty} \frac{\frac{2n^3+5n^2}{n^4-3n^3+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^4+5n^3}{n^4-3n^3+2}\cdot\frac{1/n^4}{1/n^4} = \lim_{n\to\infty} \frac{2+5/n}{1-3/n+2/n^4} = 2$$

since $C>0$ finite

the series both diverge

O.K. to skip these details

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{2n}$

$$\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{2n}\cdot\frac{1/\sqrt{n^2}}{1/\sqrt{n^2}} = \lim_{n\to\infty} \frac{\sqrt{1+1/n^2}}{2} = \frac{1}{2} \neq 0$$

Series diverges by test for divergence

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(5n)^n}{n^{3n}}$

Root test $\rho = \lim_{n\to\infty} \sqrt[n]{\frac{(5n)^n}{n^{3n}}} = \lim_{n\to\infty} \left(\frac{5n}{n^3}\right) = 0 < 1$

So series converges absolutely by root test.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$

series does not converge absolutely since

$\ln(n+1) < n+1$

$\frac{1}{\ln(n+1)} > \frac{1}{n+1}$ and $\sum \frac{1}{n+1}$ diverges

limit comparison $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = 1$

And since $\sum \frac{1}{n}$ div. then $\sum \frac{1}{n+1}$ div.

So $\sum \frac{1}{\ln(n+1)}$ diverges by comparison

But by AST, given series does converge:

$\frac{1}{\ln(n+1)}$ decreasing for all n

$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$

So series is conditionally convergent

(e) $\sum_{n=1}^{\infty} \frac{2^n}{(2n)!}$ Ratio test

$\lim_{n\to\infty} \left(\frac{2^{n+1}}{(2(n+1))!} \frac{(2n)!}{2^n} \right) = \lim_{n\to\infty} \frac{2\,(2n)!}{(2n+2)(2n+1)(2n)!}$

$= \lim_{n\to\infty} \frac{2}{(2n+2)(2n+1)} = 0 < 1$

So series converges absolutely by the ratio test

(10) Given the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (20 points)

(a) Show that the given series satisfies the conditions of the integral test.
(b) Use the integral test to show that the series converges.
(c) Approximate the sum of the series by using the sum of the first 10 terms.
(d) Estimate the error involved in the above approximation.
(e) Determine how many terms would need to be added to obtain error < 0.005.
(f) Use $s_n + \int_{n+1}^{\infty} f(x)dx \le S \le s_n + \int_{n}^{\infty} f(x)dx$ with n=10 to obtain a better estimate of the sum of the series.
(g) Estimate the error in using the approximation to S from part (f).

a) Consider $f(x) = \frac{1}{x^2}$: Conts. $x > 0$, Positive $x>0$, decreasing $x>0$ $\Rightarrow$ $\sum \frac{1}{n^2}$ Satisfies reqmts. of integral test

b) $\int_1^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty} -\frac{1}{x}\Big]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1.$ So integral converges thus series converges

c) $S_{10} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{10^2} \approx 1.54977$

d) $R_n \le \int_n^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{n}\right) = \frac{1}{n}$

So $R_{10} \le \frac{1}{10}$

e) $R_n \le \frac{1}{n} < .005$ need

$\frac{1}{n} < \frac{5}{1000} \Rightarrow n > 200$ 201 terms

f) $S_{10} + \frac{1}{11} \le S \le S_{10} + \frac{1}{10}$

Best estimate at midpoint $S \approx \frac{2S_{10} + \frac{1}{11} + \frac{1}{10}}{2} = 1.6452$

g) Error $\le \frac{1}{2}$ length of interval $= \frac{\frac{1}{10} - \frac{1}{11}}{2} = \frac{1}{220} \approx .0045$