(1) Given the following matrices: (20 points)

$A = \begin{bmatrix} 6 & 1 \\ 3 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ 3 & -4 & 3 \\ 1 & -2 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 1 & -7 \\ 2 & -5 \end{bmatrix}$ $D = \begin{bmatrix} -4 & 0 & -2 \\ 3 & 1 & -7 \end{bmatrix}$ $E = \begin{bmatrix} -1 & -2 & 0 & 4 \\ 4 & -1 & 3 & 0 \\ 7 & 2 & -1 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix}$ Find the following, if possible. (If not possible, say so.)

(a) DA not possible

(b) A + C $\begin{bmatrix} 7 & -6 \\ 5 & -7 \end{bmatrix}$

(c) A C $\begin{bmatrix} 8 & -47 \\ -1 & -31 \end{bmatrix}$

(d) DB $\begin{bmatrix} -4 & 0 & -2 \\ 3 & 1 & -7 \end{bmatrix} \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ 3 & -4 & 3 \\ 1 & -2 & 3 \end{bmatrix} =$

$$\begin{bmatrix} -2 & 8 & -8 \\ -4 & 7 & -\frac{33}{2} \end{bmatrix}$$

(g) B^{-1}
(answer here, work on next page)

$$B^{-1} = \begin{bmatrix} -6/5 & 2/5 & -1/5 \\ -6/5 & -1/10 & 3/10 \\ -2/5 & -1/5 & 3/5 \end{bmatrix}$$

(h) det (E)
(answer here, work on next page)

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Work for #1 (g) and (h)

(g) $B = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ 3 & -4 & 3 \\ 1 & -2 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc:ccc} 0 & -1 & \frac{1}{2} & 1 & 0 & 0 \\ 3 & -4 & 3 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 \leftrightarrow R3} \left[\begin{array}{ccc:ccc} 1 & -2 & 3 & 0 & 0 & 1 \\ 3 & -4 & 3 & 0 & 1 & 0 \\ 0 & -1 & \frac{1}{2} & 1 & 0 & 0 \end{array}\right]$$

$$-3R1+R2 \to R2 \left[\begin{array}{ccc:ccc} 1 & -2 & 3 & 0 & 0 & 1 \\ 0 & 2 & -6 & 0 & 1 & -3 \\ 0 & -1 & \frac{1}{2} & 1 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc:ccc} 1 & -2 & 3 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1/2 & -3/2 \\ 0 & -1 & \frac{1}{2} & 1 & 0 & 0 \end{array}\right]$$

$$R_2+R_3 \to R_3 \left[\begin{array}{ccc:ccc} 1 & -2 & 3 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1/2 & -3/2 \\ 0 & 0 & -\frac{5}{2} & 1 & \frac{1}{2} & \frac{-3}{2} \end{array}\right] \xrightarrow[-\frac{2}{5}R_3 \to R_3]{} \left[\begin{array}{ccc:ccc} 1 & -2 & 3 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 1/2 & -3/2 \\ 0 & 0 & 1 & -\frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \end{array}\right]$$

$$\begin{array}{l} -3R3+R1 \to R1 \\ 3R3+R2 \to R2 \end{array} \left[\begin{array}{ccc:ccc} 1 & -2 & 0 & \frac{6}{5} & \frac{3}{5} & \frac{-4}{5} \\ 0 & 1 & 0 & -\frac{6}{5} & -\frac{1}{10} & \frac{3}{10} \\ 0 & 0 & 1 & -\frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \end{array}\right] \xrightarrow[2R2+R1 \to R1]{} \left[\begin{array}{ccc:c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -6/5 & 2/5 & -1/5 \\ -6/5 & -1/10 & 3/10 \\ -2/5 & -1/5 & 3/5 \end{matrix}}_{B^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

(h) $\det(E) = \begin{vmatrix} -1 & -2 & 0 & 4 \\ 4 & -1 & 3 & 0 \\ 7 & 2 & -1 & 1 \\ 2 & 1 & 0 & 0 \end{vmatrix}$

minus position

$$= -2 \begin{vmatrix} -2 & 0 & 4 \\ -1 & 3 & 0 \\ 2 & -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 0 & 4 \\ 4 & 3 & 0 \\ 7 & -1 & 1 \end{vmatrix}$$

$$= -2\left[-2(3)+4(-5)\right] + 1\left[-1(3)+4(-25)\right]$$

$$= -2\left[-26\right] + 1\left[-103\right]$$

$$= 52 - 103$$

$$= -51$$

(2) Use Cramer's Rule to solve the following system. $\begin{cases} \frac{1}{3}x - 2y = 6 \\ \frac{2}{5}x - y = 3 \end{cases}$ (10 points)

$D = \begin{vmatrix} 1/3 & -2 \\ 2/5 & -1 \end{vmatrix} = -\frac{1}{3} + \frac{4}{5} = \frac{+7}{15}$

$D_x = \begin{vmatrix} 6 & -2 \\ 3 & -1 \end{vmatrix} = 0$

$D_y = \begin{vmatrix} 1/3 & 6 \\ 2/5 & 3 \end{vmatrix} = 1 - \frac{12}{5} = \frac{-7}{5}$

$x = \frac{0}{+7/15} = 0$

$y = \frac{-7/5}{+7/15} = -3$

$(0, -3)$

(3) Express the system of linear equations as a matrix equation of the form AX=B. Then solve the matrix equation by multiplying each side by the inverse of the coefficient matrix.

(14 points)

$\begin{cases} 2x + 4y = 1 \\ x - 3y = 4 \end{cases}$

$\begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/10 & 4/10 \\ 1/10 & \frac{-2}{10} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{19}{10} \\ -\frac{7}{10} \end{bmatrix}$

Find inverse of $A \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$

$A^{-1} = \frac{-1}{10} \begin{bmatrix} -3 & -4 \\ -1 & +2 \end{bmatrix} = \begin{bmatrix} 3/10 & 4/10 \\ 1/10 & -2/10 \end{bmatrix}$

(4) Solve: $\begin{cases} 2x^2 + 4y = 13 \\ x^2 - y^2 = \frac{7}{2} \end{cases}$ (14 points)

Eq1 + −2Eq2 $\quad 4y + 2y^2 = 13 - 7$

$2y^2 + 4y - 6 = 0$

$2(y^2 + 2y - 3) = 0$

$2(y+3)(y-1) = 0$

$y = -3 \qquad y = 1$

$2x^2 = 13 - 4y$

$2x^2 = 25 \qquad 2x^2 = 9$

$x^2 = \frac{25}{2} \qquad x^2 = \frac{9}{2}$

$x = \pm 5/\sqrt{2} \qquad x = \pm 3/\sqrt{2}$

$\left(\pm\frac{5}{\sqrt{2}}, -3\right)\left(\pm\frac{3}{\sqrt{2}}, 1\right)$

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(5) Solve using any of the methods discussed in class. (14 points)

$\begin{cases} 2x - y + z = 4 \\ x + 3y + 2z = -1 \\ 7x \quad + 5z = 11 \end{cases}$ $\quad \begin{bmatrix} 2 & -1 & 1 & 4 \\ 1 & 3 & 2 & -1 \\ 7 & 0 & 5 & 11 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 3 & 2 & -1 \\ 2 & -1 & 1 & 4 \\ 7 & 0 & 5 & 11 \end{bmatrix}$ $\begin{matrix} -2R1 + R2 \to R2 \\ -7R1 + R3 \to R3 \end{matrix}$

$\begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & -7 & -3 & 6 \\ 0 & -21 & -9 & 18 \end{bmatrix} \xrightarrow[-3R2 + R3 \to R3]{} \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & -7 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ← row of zeros means dependent case... infinitely many solutions.

$\begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 1 & \frac{3}{7} & -\frac{6}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad x + 3y + 2z = -1$

$y + \frac{3}{7}z = -\frac{6}{7}$

Let $z = t$.

Then $y = -\frac{6}{7} - \frac{3}{7}t$

and $x = -3y - 2z - 1$

$= -3\left(-\frac{6}{7} - \frac{3}{7}t\right) - 2t - 1$

$= \frac{18}{7} + \frac{9}{7}t - 2t - 1$

$= -\frac{5}{7}t + \frac{11}{7}$

$\left(\frac{11}{7} - \frac{5}{7}t,\ -\frac{6}{7} - \frac{3}{7}t,\ t\right)$

(6) Find the partial fraction decomposition of $\dfrac{x^2-12x+4}{x^2(x^2+4)}$ (14 points)

$$\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+4}=\frac{x^2-12x+4}{x^2(x^2+4)}$$

$\cancel{Ax^2(Cx+D)+B}$

$$Ax(x^2+4)+B(x^2+4)+(Cx+D)x^2=x^2-12x+4$$

$Ax^3 \quad + 4Ax$

$Bx^2 \quad + 4B$

$Cx^3 + Dx^2$

$$(A+C)x^3+(B+D)x^2+4Ax+4B=x^2-12x+4$$

$$\begin{cases} A + C = 0 \\ B + D = 1 \\ 4A = -12 \\ 4B = 4 \end{cases}$$

$C=3$

$D=0$

$A=-3$

$B=1$

$$\frac{-3}{x}+\frac{1}{x^2}+\frac{3x}{x^2+4}$$

check by combining

(7) Use matrix methods (Gaussian elimination or Gauss Jordan) to solve: (14 points)

$-x - 2y - z = -3$
$2x + y + z = 16$
$x + y + 2z = 9$

You must obtain row echelon form or reduced row echelon form. Be sure to label operations performed at each step.

$$\begin{bmatrix} -1 & -2 & -1 & -3 \\ 2 & 1 & 1 & 16 \\ 1 & 1 & 2 & 9 \end{bmatrix} \xrightarrow[\substack{2R_1+R_2\to R_2 \\ R_1+R_3\to R_3}]{-1R_1\to R_1} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -3 & -1 & 10 \\ 0 & -1 & 1 & 6 \end{bmatrix} \xrightarrow[R_2\leftrightarrow R_3]{-R_3\to R_3} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -6 \\ 0 & -3 & -1 & 10 \end{bmatrix}$$

$$\begin{matrix} 3R_2+R_3 \\ \to R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & -4 & -8 \end{bmatrix} \xrightarrow[-\frac{1}{4}R_3\to R_3]{} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_2+R_3\to R_2]{-R_3+R_1\to R_1} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Can stop here for Gaussian elimination

$$-2R_2+R_1\to R_1 \quad \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$(9, -4, 2)$